J Glob Optim (2010) 47:161–171
DOI 10.1007/s10898-009-9463-6

The unit-level-core for multi-choice games: the replicated core for TU games

Yan-An Hwang · Yu-Hsien Liao

Received: 14 December 2008 / Accepted: 27 July 2009 / Published online: 8 August 2009

Abstract This note extends the solution concept of the core for traditional transferable-utility (TU) games to multi-choice TU games, which we name the unit-level-core. It turns out that the unit-level-core of a multi-choice TU game is a "replicated subset" of the core of a corresponding "replicated" TU game. We propose an extension of the theorem of Bondareva (Probl Kybern 10:119–139, 1963) and Shapley (Nav Res Logist Q 14:453–460, 1967) to multi-choice games. Also, we introduce the reduced games for multi-choice TU games and provide an axiomatization of the unit-level-core on multi-choice TU games by means of consistency and its converse.

Keywords Multi-choice TU games · Replicated TU games · Unit-level-core · Reduced games

1 Introduction

A *multi-choice transferable-utility (TU) game*, introduced by Hsiao and Raghavan [9], is a generalization of a traditional TU game. In a traditional TU game, each player is either fully involved or not involved at all in participation with some other agents, while in a multi-choice TU game, each player is allowed to participate with finite many different activity levels. As we knew, solutions on multi-choice TU games could be applied in many fields such as economics, political sciences, accounting, and even military sciences.

The core is, perhaps, the most intuitive solution concept in game theory. Two extensions of the core on multi-choice TU games were proposed by Hwang and Liao [12], and van den Nouweland et al. [17], respectively. Here we offer a different extension of the core on

Y.-A. Hwang
Department of Applied Mathematics, National Dong Hwa University, Hualien 974, Taiwan
e-mail: yahwang@mail.ndhu.edu.tw

Y.-H. Liao (✉)
Department of Applied Mathematics, National Pingtung University of Education, Pingtung 900, Taiwan
e-mail: d9211001@ems.ndhu.edu.tw

multi-choice TU games, and we name it the *unit-level-core*. The unit-level-core is based on the definition of the core on both fuzzy games (Aubin [1,2]) and continuum market games (see Section 5.2, Hart [7]). It is also similar to the Aumann–Shapley method proposed by Moulin [16]. Based on the notion of "replicated game", Calvo and Santos [4] applied this method to multi-choice TU games in particular. And they showed that the extension of the Shapley value on multi-choice TU games proposed by van den Nouweland et al. [17] coincides with the solution proposed by Moulin [16]. In this paper, we point out that the unit-level-core of a multi-choice TU game is a "replicated subset" of the core of a corresponding "replicated" TU game.

One of interesting applications of traditional TU games has been done in the setting of "price problem". This kind of problem can be formulated as follows: let $N = \{1, 2, \ldots, n\}$ be a set of products, goods or services that can be provided jointly by some organizations. Let $c(S)$ be the profit of providing the items in S jointly, for each subset $S \subseteq N$. The function c is called a profit function. Modelled in this way, a price problem can be considered as a cooperative game, with c being its characteristic function. The core provides "equilibrium prices" for pricing between products (or goods), in the sense that this organization can get profit from each combination of these goods under an equilibrium price system.

Here we assume that "commodities are indivisible goods (cars, machines, buildings, etc.)" and this family of indivisible goods are only available in finite number of levels. This is the kind of situations that we want to cover in this paper: price problem for which products can be provided at a certain finite number of levels. A multi-choice TU game is exactly an appropriate game-theoretic tool for modelling this setting. Initially, we introduce the notion "the per unit level payoff vector". It differs from the standard definition of a payoff vector in the context of multi-choice TU games. Each component of a payoff vector represents the payoff to player corresponding to his activity level; however, each component of a unit level payoff vector represents the per unit level payoff that player receives. Thus, each component of a unit level payoff vector means the per unit price of a particular good. And the unit-level-core of a multi-choice TU game exactly provides the per unit equilibrium prices between goods.

van den Nouweland et al. [17] referred to other applications of multi-choice games, such as a large building project with a deadline and a penalty for every day if this deadline is overtime. The date of completion depends on the effort of how all of the people plunged into the project: the greater they exert themselves, the sooner the project will be completed. This situation gives rise to a multi-choice game. The worth of a coalition where each player works at a certain activity level is defined as the minus of the penalty which needs to be paid for giving the date of completion of the project when every player makes the relative effort. Another application appears in a large company with many divisions, where the profit-making depends on their performance. This gives rise to a multi-choice game in which the players are the divisions and the worth of a coalition where each division functions at a certain level is the corresponding profit made by the company.

In Sect. 4, we offer an extension of the Bondareva–Shapley theorem, which is a necessary and sufficient condition for the non-emptiness of the unit-level-core of a multi-choice TU game. In Sect. 6, inspired by Serrano and Volij [22], we show that the unit-level-core is the only solution satisfying one-person rationality, individual rationality, consistency and converse consistency.

2 Preliminaries

Let U be the universe of players. Suppose each player i has $m_i \in \mathbb{N}$ levels at which he can actively participate. Let $m^U \in \mathbb{N}^U$ be the vector that describes the number of activity levels for each player, in which he can actively participate. Let $N \subseteq U$ be a set of players. Denote $m^U_N \in \mathbb{R}^N$ to be the restriction of m^U to N. For $i \in U$, we set $M_i = \{0, 1, \ldots, m_i\}$ as the action space of player i, where the action 0 means not participating. For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multi-choice TU game** is a pair (N, v),[1] where N is a non-empty and finite set of players and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $\alpha = (\alpha_i)_{i \in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $\alpha_i \in M_i$ with $v(0_N) = 0$. Denote the class of all multi-choice TU games by $\mathcal{MC}$.

Given $(N, v) \in \mathcal{MC}$, $\alpha \in M^N$ and $S \subseteq N$, we denote $\alpha_S \in \mathbb{R}^S$ to be the restriction of α to S. Furthermore, we let $|S|$ be the number of elements in S, and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e^S_i(N)$ satisfies

$$e^S_i(N) = \begin{cases} 1 & \text{if } i \in S\,, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if no confusion can arise $e^S(N)$ will be denoted by e^S.

Given $(N, v) \in \mathcal{MC}$. A **unit level payoff vector** of (N, v) is a function $x : N \to \mathbb{R}$ where the number x_i represents the per unit level payoff that player i receives for all $i \in N$, hence $m_i \cdot x_i$ is the total payoff that player i receives at (N, v). For simplicity, "payoff vector" instead of "unit level payoff vector" if there is no danger of confusion.[2] Given $(N, v) \in \mathcal{MC}$, $x \in \mathbb{R}^N$, $\alpha \in M^N$ and $S \subseteq N$, we denote $x_S \in \mathbb{R}^S$ to be the restriction of x to S, $x(S) = \sum_{i \in S} x_i$, and $x(\alpha) = \sum_{i \in N} \alpha_i \cdot x_i$. Then

- a payoff vector x of $(N, v) \in \mathcal{MC}$ is **efficient (EFF)** if $x(m) = v(m)$
- a payoff vector x of $(N, v) \in \mathcal{MC}$ is **individually rational (IR)** if for all $i \in N$ and for all $j \in M_i$, $j \cdot x_i \geq v(je^{\{i\}})$.

Moreover, x is an **imputation** of (N, v) if it is EFF and IR. And the set of **feasible payoff vectors** of (N, v) is denoted by

$$X^*(N, v) = \{x \in \mathbb{R}^N \,|\, x(m) \leq v(m)\},$$

whereas

$$X(N, v) = \{\, x \in \mathbb{R}^N \,|x \quad \text{is EFF}\,\}$$

is the set of **preimputations** of (N, v) and the set of imputations of (N, v) is denoted by

$$I(N, v) = \{\, x \in \mathbb{R}^N \,|x \quad \text{is an imputation of} \quad (N, v)\,\}.$$

Definition 1 The **unit-level-core** $C(N, v)$ of $(N, v) \in \mathcal{MC}$ consists of all $x \in X(N, v)$ that satisfy for all $\alpha \in M^N$, $x(\alpha) \geq v(\alpha)$, i.e.,

$$C(N, v) = \{x \in X(N, v) \mid x(\alpha) \geq v(\alpha) \text{ for all } \alpha \in M^N\}.$$

[1] As $m^U \in \mathbb{N}^U$ is fixed throughout the note, we write (N, v) rather than (N, m^U_N, v). For convenience, if no confusion can arise m^U_N will be denoted by m throughout the note.

[2] Formally, the definition of a payoff vector should be as follows. Denote $M = \{(i, j) \mid i \in N, j \in M_i\}$. A **payoff vector** of (N, v) is a function $x : M \to \mathbb{R}$, where, for all $i \in N$ and $j \in M_i \setminus \{0\}$, x_{ij} denotes the increase in payoff to player i corresponding to a change of activity from level $j - 1$ to level j by this player and $x_{i0} = 0$ for all $i \in N$.

3 Replication

In this section, we point out the unit-level-core of a multi-choice TU game is a "replicated subset" of the core of a corresponding "replicated" TU game. To avoid confusion, a multi-choice TU game is denoted by $(N, v_{\mathcal{MC}})$ at the moment. A **coalitional traditional game with transferable utility (traditional TU game)** is a pair (N, v_{TU}) where N is a coalition and v_{TU} is a mapping such that $v_{TU} : 2^N \longrightarrow \mathbb{R}$ and $v_{TU}(\emptyset) = 0$.

The **core** $C_{TU}(N, v_{TU})$ of a traditional TU game (N, v_{TU}) is defined by

$$C_{TU}(N, v_{TU}) = \{x \in \mathbb{R}^N | x(N) = v_{TU}(N) \text{ and for all } \quad S \subseteq N, x(S) \geq v_{TU}(S)\}.$$

Given a multi-choice TU game $(N, v_{\mathcal{MC}})$. Let N^m be a set of **replicated players** defined as:

$$N^m = \bigcup_{i \in N} N_i^m,$$

where for all $i \in N$, $N_i^m = \{i_1, \ldots, i_{m_i}\}$. Now, for any $T \subseteq N^m$, we define the action vector $\alpha(T) \in M^N$ by for all $i \in N$,

$$\alpha_i(T) = |T \cap N_i^m|.$$

Then, we define the **replicated TU game** $\left(N^m, v_{TU}^m\right)$ by for all $T \subseteq N^m$,

$$v_{TU}^m(T) = v_{\mathcal{MC}}[\alpha(T)].$$

The next result shows that the unit-level-core of a multi-choice TU game is a "replicated subset" of the core of a corresponding replicated TU game.

Proposition 1 *Let* $(N, v_{\mathcal{MC}})$ *be a multi-choice TU game and* $\left(N^m, v_{TU}^m\right)$ *be the corresponding replicated TU game. If* $x = (x_i)_{i \in N} \in C(N, v_{\mathcal{MC}})$*, then* $\mathbf{x} = \left((x_{i_k})_{i_k \in N_i^m}\right)_{i \in N} \in C_{TU}\left(N^m, v_{TU}^m\right)$*, where* $x_{i_k} = x_i$ *for all* $i_k \in N_i^m$ *and for all* $i \in N$.

Proof The proof is straightforward, hence, we omit it. □

4 Balancedness

In the section we propose an extension of the theorem of Bondareva [3] and Shapley [23] to multi-choice TU games. We start with the following definition.

Definition 2 A multi-choice TU game (N, v) is **balanced** if for all maps $\lambda : M^N \rightarrow \mathbb{R}_+$ satisfying for all $i \in N$,

$$\sum_{\alpha \in M^N} \alpha_i \cdot \lambda(\alpha) = m_i, \tag{1}$$

it holds that $\sum_{\alpha \in M^N} \lambda(\alpha) \cdot v(\alpha) \leq v(m)$.

Theorem 1 *Let* $(N, v) \in \mathcal{MC}$*. The unit-level-core* $C(N, v)$ *is non-empty if and only if* (N, v) *is balanced.*

Proof It is clear that $C(N, v) \neq \emptyset$ if and only if there exists x such that

$$x(m) = v(m), \tag{2}$$

and for all $\alpha \in M^N$,

$$x(\alpha) \geq v(\alpha). \tag{3}$$

Let $X = \{x \mid \text{for all } \alpha \in M^N,\ x(\alpha) \geq v(\alpha)\}$. Then, there exists x satisfies conditions (2) and (3) if and only if there exists $x^* \in X$ such that

$$v(m) = \sum_{i \in N} m_i \cdot x_i^* = x^*(m) = \min_{x \in X} \left\{ \sum_{i \in N} m_i \cdot x_i \right\}. \tag{4}$$

Let $\Lambda = \{\lambda \mid \lambda : M^N \to \mathbb{R}_+ \text{ and } \lambda \text{ satisfies (1)}\}$. From the duality theorem of linear programming theory, condition (4) is equivalent to

$$v(m) = \max_{\lambda \in \Lambda} \left\{ \sum_{\alpha \in M^N} \lambda(\alpha) \cdot v(\alpha) \right\}.$$

Hence (N, v) is balanced. □

5 Axioms and reduced game

A **solution on** $\mathcal{MC}$ is a function σ which associates with each $(N, v) \in \mathcal{MC}$ a subset $\sigma(N, v)$ of $X^*(N, v)$. We will make use of the following axioms:

- **Efficiency (EFF)**: For all $(N, v) \in \mathcal{MC}$, $\sigma(N, v) \subseteq X(N, v)$.
- **Individual rationality (IR)**: For all $(N, v) \in \mathcal{MC}$, for all $x \in \sigma(N, v)$, for all $i \in N$ and for all $j \in M_i$, $j \cdot x_i \geq v(je^{\{i\}})$.
- **One-person rationality (OPR)**: For all $(N, v) \in \mathcal{MC}$ with $|N| = 1, \sigma(N, v) = I(N, v)$.

We extend to the multi-choice TU games case the reduced game introduced by Davis and Maschler [5].

Definition 3 Let $(N, v) \in \mathcal{MC}$, $x \in X^*(N, v)$ and $S \subseteq N$, $S \neq \emptyset$. The **reduced game with respect to** S **and** x is the game[3] $(S, v_{S,x})$ where

$$v_{S,x}(\alpha) = \begin{cases} 0 & \alpha = 0_S, \\ v(m) - \sum_{i \in N \setminus S} m_i \cdot x_i & \alpha = m_S, \\ \max_{\beta \in M^{N \setminus S}} \{v(\alpha, \beta) - \sum_{i \in N \setminus S} \beta_i \cdot x_i\} & \text{otherwise.} \end{cases}$$

Consistency requires that if x is prescribed by σ for a game (N, v), then the projection of x to S should be prescribed by σ for the reduced game with respect to S and x for all S. Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game.

- **Consistency (CON)**: For all $(N, v) \in \mathcal{MC}$, for all $S \subseteq N$ with $S \neq \emptyset$, and for all $x \in \sigma(N, v)$, $x_S \in \sigma(S, v_{S,x})$.

[3] A reduced multi-choice TU game is always a multi-choice TU game.

Converse consistency requires that if the projection of an imputation[4] x to every proper S is consistent with the expectations of the members of S as reflected by their reduced game then x itself should be recommended for whole game.

- **Converse consistency (CCON)**: For all $(N, v) \in \mathcal{MC}$ with $|N| \geq 2$ and for all $x \in I(N, v)$, if for all $S \subset N$ such that $0 < |S| < |N|$, $x_S \in \sigma(S, v_{S,x})$, then $x \in \sigma(N, v)$.

Remark 1 Let (N, v) be a multi-choice TU game and $\left(N^m, v^m_{TU}\right)$ be the corresponding replicated TU game. For $S \subseteq N$, let $N^m_S = \cup_{i\in S} N^m_i$ be the replicated coalition of S. For $x = (x_i)_{i\in N}$, let $\mathbf{x} = \left((x_{i_k})_{i_k \in N^m_i}\right)_{i\in N}$, where $x_{i_k} = x_i$ for all $i_k \in N^m_i$ and for all $i \in N$, be the replicated vector of x. Let $(S, v_{S,x})$ be a reduced game of a multi-choice TU game (N, v) with respect to S and x, and let $\left(N^m_S, (v_{S,x})^m_{TU}\right)$ be the corresponding replicated TU game of $(S, v_{S,x})$. It is intuitive that $\left(N^m_S, (v_{S,x})^m_{TU}\right)$ coincides with $\left(N^m_S, (v^m_{TU})_{N^m_S,\mathbf{x}}\right)$, where $\left(N^m_S, (v^m_{TU})_{N^m_S,\mathbf{x}}\right)$ is the reduced game of $\left(N^m, v^m_{TU}\right)$ with respect to N^m_S and $\mathbf{x}$. Hence, in CON, if it is true that for some $x \in \sigma(N, v)$ with $x_S \in \sigma(S, v_{S,x})$, then we only expect that for the replicated vector of x, $\mathbf{x} \in \sigma\left(N^m, v^m_{TU}\right)$ with $\mathbf{x}_{N^m_S} \in \left(N^m_S, (v^m_{TU})_{N^m_S,\mathbf{x}}\right)$. The same situation happens in CCON. This is why the axiomatization does not yield the entire core of the replicated TU game, but captures only its equal-treatment allocations.

6 Axiomatization

In this section we shall use OPR, IR, CON and CCON to characterize the unit-level-core.

Lemma 1 *Let $(N, v) \in \mathcal{MC}$, $x \in C(N, v)$ and $S \subseteq N$, $S \neq \emptyset$. Then for all $\alpha \in M^S$, $\alpha \neq 0_S$,*

$$\sum_{i\in S} \alpha_i x_i \geq \max\left\{ v(\alpha, \beta) - \sum_{i\in N\setminus S} \beta_i x_i \mid \beta \in M^{N\setminus S} \right\}.$$

Proof Let $(N, v) \in \mathcal{MC}$, $x \in C(N, v)$ and $S \subseteq N$ with $S \neq \emptyset$. Let $\alpha \in M^S$, $\alpha \neq 0_S$. Then

$$\begin{aligned} &\max\left\{ v(\alpha, \beta) - \sum_{i\in N\setminus S} \beta_i x_i \mid \beta \in M^{N\setminus S} \right\} - \sum_{i\in S} \alpha_i x_i \\ &= \max\left\{ v(\alpha, \beta) - \sum_{i\in N\setminus S} \beta_i x_i - \sum_{i\in S} \alpha_i x_i \mid \beta \in M^{N\setminus S} \right\} \\ &\leq \max\{v(\gamma) - x(\gamma) \mid \gamma \in M^N\} \\ &\leq 0. \text{ (Since } \; x \in C(N, v)) \end{aligned}$$

Hence, $\sum_{i\in S} \alpha_i x_i \geq \max\left\{ v(\alpha, \beta) - \sum_{i\in N\setminus S} \beta_i x_i \mid \beta \in M^{N\setminus S} \right\}$. □

[4] In general, converse consistency only requires that the payoff vector is feasible. Here, a variant of converse consistency is obtained by strengthening its hypothesis to the requirement that the payoff vector is an imputation.

Lemma 2 *The unit-level-core satisfies consistency.*

Proof Let $(N, v) \in \mathcal{MC}$. Let $x \in C(N, v)$ and $S \subseteq N$ with $S \neq \emptyset$. Clearly, $(S, v_{S,x}) \in \mathcal{MC}$. Since $x \in C(N, v)$, x is EFF in (N, v). Hence, x_S is EFF in the reduced game $(S, v_{S,x})$ by the definition of $v_{S,x}(S)$.

It remains to show that for all $\alpha \in M^S$, $\alpha \neq m_S$, $\sum_{i \in S} \alpha_i x_i \geq v_{S,x}(\alpha)$. It is trivial if $\alpha = 0_S$.

Let $\alpha \in M^S$, $\alpha \neq m_S$, $\alpha \neq 0_S$. Then

$$
\begin{aligned}
& v_{S,x}(\alpha) - \sum_{i \in S} \alpha_i x_i \\
&\quad = \max \left\{ v(\alpha, \beta) - \sum_{i \in N \setminus S} \beta_i x_i \mid \beta \in M^{N \setminus S} \right\} - \sum_{i \in S} \alpha_i x_i .
\end{aligned}
$$

By Lemma 1, $\sum_{i \in S} \alpha_i x_i \geq v_{S,x}(\alpha)$.

So if $x \in C(N, v)$, for all $S \subseteq N$ with $S \neq \emptyset$, $x_S \in C(S, v_{S,x})$. □

Lemma 3 *The unit-level-core satisfies converse consistency.*

Proof Let $(N, v) \in \mathcal{MC}$ with $|N| \geq 2$ and let $x \in I(N, v)$. Suppose for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_{S,x}) \in \mathcal{MC}$ and $x_S \in C(S, v_{S,x})$. We will show that $x \in C(N, v)$, i.e., for all $\alpha \in M^N$, $\sum_{i \in N} \alpha_i x_i \geq v(\alpha)$. Two cases can be distinguished:

Case 1 $|N| = 2$:
Suppose that $N = \{i, j\}$. Let $\alpha = (\alpha_i, \alpha_j) \in M^N$.

- Consider $\alpha = (\alpha_i, \alpha_j) \in M^N$ with $\alpha_k = 0$ or $\alpha_k = m_k$ where $k = i, j$. Since $x \in I(N, v)$, $\sum_{k \in N} \alpha_k x_k \geq v(\alpha)$.
- Consider $\alpha = (\alpha_i, \alpha_j) \in M^N$ with $\alpha_i \neq m_i$. Then

$$
\begin{aligned}
\alpha_i x_i &\geq v_{\{i\},x}(\alpha_i) \ \big(\text{By} \quad x_i \in C(\{i\}, v_{\{i\},x})\big) \\
&= \max\{v(\alpha_i, l) - l x_j \mid l \in M_j\} \ \big(\text{By the definition of} \quad v_{\{i\},x}(\alpha_i)\big) \\
&\geq v(\alpha_i, \alpha_j) - \alpha_j x_j . \quad \big(\text{Take} \quad l = \alpha_j\big)
\end{aligned}
$$

  Hence, $\sum_{k \in N} \alpha_k x_k \geq v(\alpha)$.
- Consider $\alpha = (\alpha_i, \alpha_j) \in M^N$ with $\alpha_j \neq m_j$. The proof is similar to the previous arguments by considering the reduced game $(\{j\}, v_{\{j\},x})$, we omit it.

Case 2 $|N| > 2$:
Let $\alpha \in M^N$, $\alpha \neq 0_N$. If $\alpha = m$, we have done by the EFF of x. Suppose $\alpha \neq m$.

- If there exists $i \in N$ such that $0 < \alpha_i < m_i$. Consider the reduced game $(\{i\}, v_{\{i\},x})$. Then

$$
\begin{aligned}
\alpha_i x_i &\geq v_{\{i\},x}(\alpha_i) \ (\text{By} \quad x_i \in C(\{i\}, v_{\{i\},x})) \\
&= \max \left\{ v(\alpha_i, \beta) - \sum_{j \in N \setminus \{i\}} \beta_j x_j \mid \beta \in M^{N \setminus \{i\}} \right\} \\
&\quad \big(\text{By the definition of} \quad v_{\{i\},x}(\alpha_i)\big) \\
&\geq v(\alpha) - \sum_{j \in N \setminus \{i\}} \alpha_j x_j . \ \big(\text{Take} \quad \beta = \alpha_{N \setminus \{i\}}\big)
\end{aligned}
$$

  Hence, $\sum_{k \in N} \alpha_k x_k \geq v(\alpha)$.

- If for all $i \in N$, $\alpha_i = 0$ or $\alpha_i = m_i$. Assume that $\alpha_i = m_i$ and $\alpha_j = 0$ where $i, j \in N$. Consider the reduced game $(\{i, j\}, v_{\{i,j\},x})$. Then

$$\begin{aligned}
\alpha_i x_i &\geq v_{\{i,j\},x}(\alpha_i, 0) \left(\text{By} \quad x_i \in C(\{i, j\}, v_{\{i,j\},x})\right) \\
&= \max \left\{ v(\alpha_i, 0, \beta) - \sum_{k \in N\setminus\{i,j\}} \beta_k x_k \mid \beta \in M^{N\setminus\{i,j\}} \right\} \\
&\quad \left(\text{By the definition of} \quad v_{\{i,j\},x}(\alpha_i, 0)\right) \\
&\geq v(\alpha) - \sum_{k \in N\setminus\{i,j\}} \alpha_k x_k \left(\text{Take} \quad \beta = \alpha_{N\setminus\{i,j\}}\right) \\
&= v(\alpha) - \sum_{k \in N\setminus\{i\}} \alpha_k x_k \left(\text{Since} \quad \alpha_j = 0\right).
\end{aligned}$$

Hence, $\sum_{k \in N} \alpha_k x_k \geq v(\alpha)$. □

The following lemma is an extension of Serrano and Volij's [22] result.

Lemma 4 *Let σ be a solution on $\mathcal{MC}$. If σ satisfies OPR and CON then it also satisfies EFF.*

Proof Assume, on the contrary, that there exist $(N, v) \in \mathcal{MC}$ and $x \in \sigma(N, v)$ such that $\sum_{j \in N} m_j x_j < v(m)$. Let $i \in N$. Consider the reduced game $(\{i\}, v_{\{i\},x})$. We see that $v_{\{i\},x}(m_i) = v(m) - \sum_{j \in N\setminus\{i\}} m_j x_j$. By CON of σ, $x_i \in \sigma(\{i\}, v_{\{i\},x})$. By OPR of σ, $m_i x_i \geq v_{\{i\},x}(m_i) = v(m) - \sum_{j \in N\setminus\{i\}} m_j x_j$. That is, $\sum_{j \in N} m_j x_j \geq v(m)$. Thus, the desired contradiction has been obtained. □

The technique of the following proof is based on the *Elevator Lemma* introduced by Thomson [26], which have been made many times in the literature.

Theorem 2 *A solution σ on $\mathcal{MC}$ satisfies OPR, IR, CON and CCON if and only if for all $(N, v) \in \mathcal{MC}$, $\sigma(N, v) = C(N, v)$.*

Proof By Lemmas 2, 3, the unit-level-core satisfies CON and CCON. And clearly, the unit-level-core satisfies OPR and IR.

To prove the uniqueness, assume that a solution σ satisfies OPR, IR, CON and CCON. By Lemma 4, σ satisfies EFF. Let$(N, v) \in \mathcal{MC}$. The proof proceeds by induction on the number $|N|$. If $|N| = 1$ then by OPR of σ, $\sigma(N, v) = I(N, v) = C(N, v)$. Assume that $\sigma(N, v) = C(N, v)$ if $|N| < k$, $k \geq 2$.

The case $|N| = k$: (This is an instance of the Elevator Lemma.)

First we prove that $\sigma(N, v) \subseteq C(N, v)$. Let $x \in \sigma(N, v)$. Since σ satisfies IR and EFF, $x \in I(N, v)$. By CON of σ, for all $S \subset N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, v_{S,x})$. By the induction hypothesis, for all $S \subset N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, v_{S,x}) = C(S, v_{S,x})$. By CCON of the unit-level-core, $x \in C(N, v)$. The opposite inclusion may be shown analogously by interchanging the roles of σ and C. Hence $\sigma(N, v) = C(N, v)$. □

The following examples show that each of the axioms used in Theorem 2 is logically independent of the others.[5]

Example 1 Let $\sigma(N, v) = \emptyset$ for all $(N, v) \in \mathcal{MC}$. Then σ satisfies IR, CON and CCON, but it violates OPR.

[5] In order to show the logical independence of the used axioms $|U| \geq 2$ is needed.

Example 2 Define the solution σ on $\mathcal{MC}$ by

$$\sigma(N, v) = \begin{cases} I(N, v) \ , & \text{if} \quad |N| = 1 \\ X(N, v) \ , & \text{otherwise.} \end{cases}$$

Then σ satisfies OPR, CON and CCON, but it violates IR.

Example 3 Let $\sigma(N, v) = I(N, v)$ for all $(N, v) \in \mathcal{MC}$. Then σ satisfies OPR, IR and CCON, but it violates CON.

Example 4 Define the solution σ on $\mathcal{MC}$ by

$$\sigma(N, v) = \begin{cases} I(N, v) \ , & \text{if} \quad |N| = 1 \\ \emptyset \ , & \text{otherwise.} \end{cases}$$

Then σ satisfies OPR, IR and CON, but it violates CCON.

7 Final remarks

Consistency was originally introduced by Harsanyi [6] under the name of bilateral equilibrium, also in the context of bargaining and, what's more, in connection with the Nash solution. This axiom has served to characterize solution concepts in game theory. Of course, as is well-known, the definition of the reduced game is the key step in this axiom. Various definitions of a reduced game depend upon exactly how are the players outside are being paid off. Roughly speaking, there are three different types of imaginary reduced games in the literature, the DM-reduced game (Davis-Maschler [5]), the HM-reduced game (Hart and Mas-Colell [8]), and the M-reduced game (Moulin [15]).

The M-reduced game is based on the idea that, when renegotiating the payoff distribution within the subgroup, any coalition has to guarantee the "original" payoffs to "all" the members outside of the subgroup; however, in the DM-reduced game, any coalition in the subgroup can choose the best group of partners in the members outside of the subgroup (instead of being forced to get together with "all" the members outside of the subgroup) provided, as before, that it pays them their "original" payoffs. According to the HM-reduced game, it does not assume that payoffs to the players outside of the subgroup are fixed. It is similar to the M-reduced game in that any coalition in the subgroup has to take along "all" the members outside of the subgroup. But the difference is that in the HM-reduced game any coalition pays them off according to the solution outcome of the appropriate subgame.

Based on the three reduced games, there are several axiomatic results in the literature:

(1) Based on the DM-reduced game: the axiomatic characterization of the prenucleolus by Sobolev [24]; Peleg [19,20] characterized the core of NTU and TU games, respectively and the prekernel of TU games; Serrano and Volij [22] axiomatized the core for the class of all economies and Hwang and Sudhölter [13] also characterized the core of NTU and TU games, respectively; Serrano and Shimomura [21] axiomatized the Nash set; Orshan and Zarzuelo [18] characterized the bilateral consistent prekernel of NTU games, and so on.

(2) Based on the HM-reduced game: the axiomatic characterization of the Shapley value by Hart and Mas-Colell [8]; Maschler and Owen [14] characterized the consistent Shapley value for Hyperplane games.

(3) Based on the M-reduced game: the axiomatic characterization of the equal allocation of nonseparable cost value by Moulin [15]; Tadenuma [25] characterized the core of NTU and TU games, respectively.

Two extensions of the Hart-Mas-Colell [8] reduction have been studied and discussed in the setting of multi-choice games by Hwang and Liao [10,11]. An extension of the Davis-Maschler [5] reduction have been studied and discussed in the setting of multi-choice games by Hwang and Liao [12]. Here, we provide another extension of the Davis-Maschler reduction. This also raises the question whether the implications of the other forms of reduction that have been discussed in the literature could also be described in our setting. In the setting of multichoice games, the Moulin [15] reduction is also the one worth considering. A natural extension of the Moulin [15] reduction to multi-choice games is as follows.

Definition 4 Let $(N, v) \in \mathcal{MC}$, $x \in \mathbb{R}^N$ and $S \subseteq N$, $S \neq \emptyset$. The **modified reduced game with respect to** S **and** x is the game $(S, v^M_{S,x})$ where

$$v^M_{S,x}(\alpha) = \begin{cases} 0 & , \quad \alpha = 0_S, \\ v(\alpha, m_{N\setminus S}) - \sum_{i \in N\setminus S} m_i x_i & , \quad \text{otherwise.} \end{cases}$$

As we have known, in the framework of TU games, Tadenuma [25] proved that on the domain of balanced games, the core is the only solution satisfying non-emptiness, individual rationality and consistency (w.r.t. Moulin reduction). However, in the framework of multi-choice games, the unit-level-core is not the only solution satisfying non-emptiness, individual rationality and consistency (w.r.t. Definition 4) on the domain of balanced multi-choice games. The reason is by way of illustration: we define a solution on the domain of balanced multi-choice games by for each balanced multi-choice game (N, v), $\sigma(N, v)$ is the set of all $x \in I(N, v)$ that satisfy for all $\alpha \in M^N$ with $\alpha_i = m_i$ for some $i \in N$, $x(\alpha) \geq v(\alpha)$. Clearly, $C(N, v) \subseteq \sigma(N, v)$ and it is not difficult to derive that σ satisfies non-emptiness, individual rationality and consistency (w.r.t. Definition 4) on the domain of balanced multi-choice games. This points out that the unit-level-core can "not" be characterized by non-emptiness, individual rationality and consistency (w.r.t. Definition 4) on the domain of balanced multi-choice games. Hence, based on the Moulin reduction, the related issue is still an open question.

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